JOURNAL OF APPROXIMATION THEORY **46**, 32–37 (1986)

Freud's Work in Constructive Function Theory

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Communicated by Paul G. Nevai

Received July 9, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

Constructive function theory seeks out the connections between the structural properties of a function and its degree of approximation. It began early in this century with the work of Jackson and Bernstein on polynomial approximation. They were interested in the error of approximation to a function $f \in C(\mathbb{T})$ by the class $\mathscr{T}_n$ of trigonometric polynomials of degree $\leqslant n$,

$$E_n^*(f) := \inf_{T \in \mathscr{T}_n} \|f - T\|_{C(\mathbb{T})}. \tag{1}$$

Weierstrass' theorem tells us that $E_n^*(f)$ decreases to zero as n tends to infinity. Jackson and Bernstein wanted to put this in a quantitative form which involves the smoothness of f, in particular its modulus of continuity:

$$\omega(f, t) := \sup_{0 < h \leqslant t} \|\Delta_h(f)\|^0_{C(\mathbb{T})}; \qquad \Delta_h(f, x) := fx(+h) - f(x). \tag{2}$$

In 1911, as part of his doctoral dissertation under Landau in Göttingen, Jackson [12] proved the now famous inequality:

$$E_n^*(f) \leqslant C\omega(, n^{-1}), \qquad n = 1, \ldots. \tag{3}$$

These were not the first estimates for E_n^* but they were the cleanest. Only a year earlier, Lebesgue had given estimates for approximation by the nth Fourier sums; but the latter contain logarithms which are absent in (3).

As natural as (3) seems, there is still the question whether it is in some sense best possible. Bernstein [2] showed that this is the case, in at least two different ways. He gave examples of functions f e.g., $f(t) = |\cos t|$, for which (3) can be reversed. In another vein, he showed that for $0 < \alpha < 1$,

0021-9045/86 \$3.00

$E_n^*(f) = O(n^{-\alpha})$ if and only if $\omega(f, t) = O(t^\alpha)$. Of course, the latter condition describes the famous Lipschitz classes, Lip α.

Conspicuously absent in the above characterization is the case $\alpha = 1$. It turns out that functions approximated with order $O(n^{-1})$ need not be in Lip 1. The mystery in this case was not settled until 1941 when Zygmund [16] showed the need for using higher order differences Δ_h^k (defined inductively) and their corresponding moduli of smoothness

$$\omega_k(f, t) := \sup_{0 < h \leqslant t} \|\Delta_h^k(f)\|_{C(\mathbb{T})}, \qquad k = 1, 2, \ldots.$$

Zygmund proved that $E_n^*(f) = O(n^{-1})$ if and only if $\omega_2(f, t) = O(t)$. This space of functions is often referred to as the Zygmund class.

Finally, early in the 1950's the whole matter of trigonometric approximation was settled by Stechkin [14]. Following Achiezer [1] (the case $k = 2$), he proved for $k = 1, 2, \ldots,$

$$E_n^*(f) \leqslant C_k \omega_k(f, n^{-1}), \qquad n = 1, 2, \ldots, \tag{4}$$

and also gave converse estimates which had the effect of characterizing for all $\alpha > 0$, the functions which are approximated with order $O(n^{-\alpha})$: they satisfy $\omega_k(f, t) = O(t^\alpha)$, $k := [\alpha] + 1$. The latter function classes are known as generalized Lipschitz spaces (Lip*α). They have many important applications in analysis.

While these investigations into trigonometric approximation had a nice clean ending, the situation regarding algebraic polynomial approximation in $C(I)$, $I := [-1, 1]$, was much different. The analogues of (4) are quite easy to prove by using what is now a standard substitution, $x = \cos\theta$, to transform the approximation of f to a matter of approximating $g(\theta) := f(\cos\theta)$ by trigonometric polynomials, which is of course solved in (4). However, such results do not characterize the Lipschitz classes. The reason for this, as was pointed out by Nikolski, is that algebraic polynomials can approximate better near the end points of the interval I. Timan [15] made this more precise by proving that there are algebraic polynomials P_n of degree $\leqslant n$ such that for $-1 \leqslant x \leqslant 1$,

$$|f(x) - P_n(x)| \leqslant C\omega(f, \Delta_n(x)), \qquad \Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}. \tag{5}$$

This added precision near the endpoints in (5) is enough to admit a converse result. In this way, we characterize Lip α, $0 < \alpha < 1$, as the set of those functions f which satisfy (5) with the right hand side replaced by $C(\Delta_n(x))^\alpha$.

But what about the case $\alpha \geqslant 1$? Zygmund's theorem suggests the need for higher order moduli of smoothness. Still, it was some eight years after

Timan's result that Freud [8] and Dzadyk [6] working independently proved

$$|f(x) - P_n(x)| \leqslant C\omega_2(f, \Delta_n(x)), \qquad -1 \leqslant x \leqslant 1, \tag{6}$$

for suitable P_n of degree n and $n = 1, 2, \ldots$. The main difficulty to be circumvented in proving (6) is that the usual substitution $x = \cos\theta$ leads to a function g whose second differences are not easily estimated by those of f.

Freud's proof of (6) is noteworthy in many respects. It is based on a two stage approximation. First, he approximates f by a continuous piecewise linear function S which interpolates f at a set of points x_k, $k = 1, \ldots, n$, which is thicker near the endpoints, thereby allowing $f - S$ to be estimated by the right side of (6). Now S can be written as a linear combination of the functions $(x - x_k)_+$. In this way, (6) is reduced to approximating these truncated powers. This is done by transforming to the trigonometric case and using standard methods of trigonometric approximation.

Brudnyi [4] had used a similar approach to Freud's in proving that (6) holds with ω_k in place of ω_2. These estimates can then be used to show that for any $\alpha > 0$, the class of functions Lip*α consists of those f such that (6) holds with the right side replaced by $C(\Delta_n(x))^\alpha$. These results completed an important chapter in the approximation of functions.

The technique of intermediate approximation by smooth functions was used often and with much success by Freud. For example, it plays a crucial role in Freud's quantitative estimates [9, 10] for approximation by a sequence (L_n) of positive linear operators on $C(I)$. Bohman and Korovkin had shown that the convergence of $L_n f$ to f for the three test functions $f(x) = 1, x, x^2$ guarantees the convergence of $L_n f$ to f for all $f \in C(I)$. Freud generalized this result and put it into a quantitative form.

Freud showed that the test functions $1, x, x^2$ can be replaced by any set ϕ_0, ϕ_1, ϕ_2 which forms an extended Chebyshev system, that is, the ϕ_i are twice continuously differentiable and each "polynomial" $\phi = c_0\phi_0 + c_1\phi_1 + c_2\phi_2$ (where not all c_k are 0) has at most two zeros in I counting multiplicity. Freud proved that if $\|\phi_i - L_n\phi_i\| = O(\lambda_n^2)$, $i = 0, 1, 2$, then for each $f \in C(I)$,

$$\|f - L_n(f)\| \leqslant C(\omega_2(f, \lambda_n) + \|f\|\, \lambda_n^2), \qquad n = 1, 2, \ldots. \tag{7}$$

To prove (7), Freud first shows that for each twice continuously differentiable function g,

$$\|g - L_n g\| \leqslant C[\|g\| + \|g''\|]\, \lambda_n^2. \tag{8}$$

When, the test functions are the power functions $1, x, x^2$, (8) follows from the expansion $g(t) = g(x) + g'(x)(t - x) + R(t, x)$ with $|R(t, x)| \leqslant$

$\frac{1}{2}\|g''\|(t-x)^2$. Indeed, since $\|g'\| \leqslant C(\|g\| + \|g''\|)$, applying L_n to this formula (at x) and using the positivity of L_n gives (8). The general case is a modification of these ideas using the properties of extended Chebyshev systems.

To deduce (7) from (8), Freud proves that for each $\delta > 0$, there is a twice differentiable g_δ such that

$$\|f-g_\delta\| + \|g''_\delta\| \leqslant C\omega_2(f, \delta). \tag{9}$$

Now, (7) follows by taking $\delta = \lambda_n$ and using (8) to estimate $\|g_\delta - L_n g_\delta\|$ and using the uniform boundedness of the L_n and (9) to estimate $\|(f-g_\delta) - L_n(f-g_\delta)\|$.

Nowadays, the above technique is encompassed in the Peetre K functional (see [5] for a discussion of the K functional in approximation). In particular, (9) is the substantial half of the proof that the K functional for interpolation between C and $C^{(2)}$ is equivalent to ω_2. For the real line or the circle, this is rather easy to prove using Steklov averages, as was shown some years earlier by Peetre [13]. On the other hand, the case of the finite interval I has certain technical difficulties since the Stecklov averages are not usable at the end points of I. Freud overcomes this by extending f to a larger interval in such a way that the modulus of continuity is not substantially increased.

There is another beautiful theorem of Freud which can likewise be reduced to the approximation of elementary functions. This one deals with one-sided approximation, namely the approximation of f by polynomials P_n of degree $\leqslant n$ which lie below f: $P_n \leqslant f$. The approximation is done in the metric of $L_1(I, d\mu)$ with $d\mu(x) := dx/\sqrt{1-x^2}$. We denote by E_n^+ the error in this type of approximation. Freud came to this problem from his study of Tauberian theorems with remainder which will be reported on by Ganelius.

If V^r denotes the set of functions f with $f^{(r-1)}$ absolutely continuous and $\text{Var}_I(f^{(r)}) = 1$, then Freud [6] proved

$$E_n^+(f) \leqslant C_r n^{-r-1}, \qquad n = 1, 2, \ldots \tag{10}$$

whenever $f \in V^r$. To approximate functions in V^r, it is enough to approximate the extreme points of V^r which are the truncated powers $(1/r!)(x-a)_+^r$, $a \in I$. From Freud's experience with orthogonal polynomials, the latter was an easy problem. The first step is to construct the Hermite interpolant Q_a to the Heaveside function $H_a(x) := (x-a)_+^0$ at the points $x_1, \ldots, x_k, x_{k+r}, \ldots, x_n$, where the x_j are the zeros of the Chebyshev polynomial C_n of degree n, and k is chosen so that $x_k < a \leqslant x_{k+1}$. That is, Q_a interpolates H_a and Q'_a interpolates H'_a at these points. Rolle's theorem shows that $0 \leqslant Q_a \leqslant 1$ on I. Hence the polynomial $P_a(x) := (x-a)^r Q_a(x)$ lies below $(x-a)^r_+$. It follows that

$$E^+_{2n}((\cdot - a)^r_+) \leqslant \int [(x-a)^r_+ - P_a(x)]\, d\mu(x) \leqslant C n^{-r-1}. \tag{11}$$

The proof of the latter inequality uses Gauss's quadrature formula to integrate P_a exactly. The ideas are very reminiscent of the classical proofs of the Chebyshev–Markov–Stieljes separation theorem.

The above theorems of Freud dealt with polynomial approximation. Freud of course looked at other aspects of approximation, some of which will surely be reported on by others. I would, however, like to say something about his work [11] with Popov on spline approximation. In the late sixties, splines were a hot topic, spurred on largely by the elegant work of Schoenberg. The constructive theory of splines was just beginning to take shape. Especially interesting was spline approximation with *free knots* since this type of approximation showed some remarkable gains in the order of approximation when compared with polynomial approximation or approximation by splines with equally spaced knots.

Freud and Popov proved that when f is a continuous function in V^r, then there is spline function $S_n \in C^{(r-1)}(I)$ which consists of n polynomial pieces, each of degree $\leqslant r$, such that

$$\|f - S_n\|_{C(I)} \leqslant C_r n^{-r-1}. \tag{12}$$

For approximation by polynomials of degree $\leqslant n$ such an order of approximation requires that $f^{(r)}$ be in the Zygmund class in $[-a, a]$, for all $a < 1$, thus in particular in every Lip α, $\alpha < 1$. We see that there is a considerable gain in the approximation order for free knot splines. When $r = 0$, this gain can be realized by choosing knots $-1 = x_0 < x_1 < \cdots < x_n = 1$ so that the variation of f is balanced on the intervals $I_j := (x_{j-1}, x_j)$: $\mathrm{Var}_{I_j} f = 1/n$, $j = 1,..., n$. The function S_n which takes on the constant value $f(x_j)$ on I_j satisfies (12).

When $r > 0$, we must work a little harder. We now choose knots so that $\mathrm{Var}_{I_j} f^{(r)} \leqslant 2/n$ and $|I_j| \leqslant 4/n$, $j = 1,..., n$. This can be realized by refining a "balanced" knot set. If ϕ_r is the function which has the constant value $f^{(r)}(x_j)$ on I_j, then

$$\|f^{(r)} = \phi_r\|_{C(I)} \leqslant 2/n. \tag{13}$$

We would like to integrate (13) to get successive approximations ϕ_j to $f(j)$, $j = r-1,..., 0$. However, simple integration will not work since the error will build up. Freud and Popov avoid this by making a small correction at each stage. Namely, they construct a spline ψ_j with knots also at the x_ν so that

$$\phi_{j-1} := f^{(j-1)}(-1) + \int_{-1}^{x} (\phi_j + \psi_j)$$

will interpolate $f^{(j-1)}$ every so often, for example, at every rth knot x_ν. This correction is made up from locally supported splines; B-splines would do quite well.

The Freud–Popov technique is useful for other types of spline approximation. For example, the knots can be fixed or monotone functions can be approximated by monontone splines. Sometime after their work, de Boor [3] developed the powerful quasi-interpolant projectors, which can usually avoid some of the technical aspects of the Freud-Popov approach. However, in problems where the approximation of f is to be approached through the approximation of its derivatives, the Freud-Popov technique still has much to offer.

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